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Heat Transfer from a Sphere Immersed in a Stream of an Inviscid Fluid at Small Péclet Number

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SUMMARY

Forced heat convection from a sphere at low Péclet number is studied using the method of matched asymptotic expansions. An inviscid theory is applied for the velocity field. The results obtained in this paper may be applicable to the flow of low Prandtl number fluids.

1. Introduction

The problem of heat transfer from a sphere immersed in a stream at small Péclet number has been studied so far by many authors. Recent contributions to the problem are made by Acrivos and Taylor [1] and by Rimmer [2], in which the Nusselt number N is expressed as an expansion in terms of the Péclet number $P=\sigma R$ using the method of matched asymptotic expansions, where σ denotes Prandtl number and R Reynolds number. No restriction on the value of σ is imposed in the former and restriction of $\sigma=O(1)$ in the latter. Since Acrivos and Taylor use the Stokes approximation of creeping flow for the velocity field, the application of their expansion to the fluid having the Prandtl number less than unity is limited to the case $P\ll\sigma$. Hence, for the flow of extremely low Prandtl number fluids such as liquid metals, the Péclet number range over which the expansion can be successfully applied is very narrow. On the other hand, Gross and Cess [3] reasoned that, when the Prandtl number becomes extremely small, an inviscid potential flow approximation can be used for the velocity in the energy equation. The experimental justification of the conjecture is given by Sajben [4] comparing the experimental results for the flow of mercury past a circular cylinder with those derived from inviscid theory [5].

In the present paper, the inviscid theory is applied to the flow around a sphere and the results may be applicable to the flow of low Prandtl number fluids. The Péclet number is assumed to be smaller than unity and the method of solution is similar to that used by Acrivos and Taylor.

2. Mathematical Formulation of the Problem

We consider a single sphere of constant temperature T_w immersed in a uniform stream of velocity U_∞ of an inviscid fluid. On choosing the co-ordinate variables as illustrated in fig. 1, the governing equation for the temperature distribution in the fluid can be written as

$$u_r'\frac{\partial t'}{\partial r'}+\frac{u_\theta'}{r'}\frac{\partial t'}{\partial\theta}=\kappa\nabla_r'^2 t', \tag{1}$$

where t' is the temperature of the fluid, κ the thermal diffusivity, u_r' and u_θ' are the velocity components in r' and θ directions and $\nabla_r'^2$ is the operator

$$\nabla_r'^2\equiv\frac{1}{r'^2}\frac{\partial}{\partial r'}\left(r'^2\frac{\partial}{\partial r'}\right)+\frac{1}{r'^2}\frac{\partial}{\partial\mu}\left\{(1-\mu^2)\frac{\partial}{\partial\mu}\right\},\qquad \mu=\cos\theta\,. \tag{2}$$

As is well known, the velocity distribution of a potential flow around a sphere is given by

$$u_r'=U_\infty\left(1-\frac{a^3}{r'^3}\right)\cos\theta\,,\quad u_\theta'=-\frac{U_\infty}{2}\left(2+\frac{a^3}{r'^3}\right)\sin\theta\,, \tag{3}$$

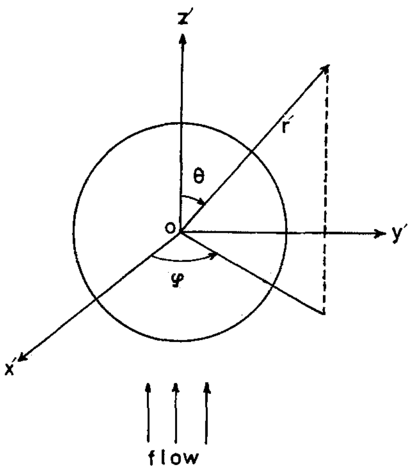

Figure 1.

where a is the radius of the sphere. The boundary conditions for the temperature field are

$$\left.\begin{aligned} t' &= T_w \quad \text{at} \quad r' = a\,, \\ t' &\to T_\infty \quad \text{as} \quad r' \to \infty\,. \end{aligned}\right\} \tag{4}$$

Defining such dimensionless quantities as

$$\left.\begin{aligned} &r = r'/a\,, \quad u_r = u'_r/U_\infty\,, \quad u_\theta = u'_\theta/U_\infty\,, \\ &t = (t' - T_\infty)/(T_w - T_\infty)\,, \quad P = U_\infty a/\kappa\,, \end{aligned}\right\} \tag{5}$$

the equations (1), (3) and (5) are written in dimensionless form as

$$u_r \frac{\partial t}{\partial r} + \frac{u_\theta}{r}\frac{\partial t}{\partial \theta} = \frac{1}{P}\nabla_r^2 t\,, \tag{6}$$

$$u_r = (1 - r^{-3})\cos\theta\,, \quad u_\theta = -\tfrac{1}{2}(2 + r^{-3})\sin\theta\,, \tag{7}$$

$$\left.\begin{aligned} t &= 1 \ \text{at} \ r = 1\,, \\ t &\to 0 \ \text{as} \ r \to \infty\,, \end{aligned}\right\} \tag{8}$$

where ∇_r^2 is a dimensionless form of $\nabla_r'^2$.

The method of solution of the eq. (6) is to obtain the temperature as an expansion in terms of Péclet number for $P < 1$ in each of two regions, one close to, and the other far from the sphere, which we call inner region and outer region respectively. In the inner region where $r = O(1)$ and the convection term is of minor importance compared with the conduction term, we assume an expansion of the form

$$t(r, \mu) = t_0(r, \mu) + f_1(P)t_1(r, \mu) + f_2(P)t_2(r, \mu) + \ldots\,, \tag{9}$$

where

$$f_{n+1}(P)/f_n(P) \to 0 \quad \text{as} \quad P \to 0\,, \tag{10}$$

which we call inner expansion. The inner expansion satisfies equation (6) and the boundary condition on the surface. In addition, the behaviour of t for $r \to \infty$ can be determined through matching with the expansion valid in the outer region, which we call outer expansion.

In the outer region, that is where $r = O(P^{-1})$, we introduce new variables defined as

$$\left.\begin{aligned} &\rho = Pr\,, \quad T(\rho, \mu) = t(r, \mu)\,, \\ &U_\rho(\rho, \mu) = u_r(r, \mu)\,, \quad U_\theta(\rho, \mu) = u_\theta(r, \mu)\,, \end{aligned}\right\} \tag{11}$$

in terms of which the energy eq. (6) becomes

$$U_\rho \frac{\partial T}{\partial \rho} + \frac{U_\theta}{\rho} \frac{\partial T}{\partial \theta} = \nabla_\rho^2 T , \tag{12}$$

where ∇_ρ^2 is the same operator as ∇_r^2 but with r replaced by ρ. This equation reflects the proper balance between the convection and conduction terms in the energy equation. The expressions for U_ρ and U_θ are obtained by rewriting eqs. (7) in the variable ρ instead of r, that is,

$$U_\rho = \cos\theta - (\cos\theta/\rho^3) P^3 , \quad U_\theta = -\sin\theta - (\sin\theta/2\rho^3) P^3 . \tag{13}$$

The solution of eq. (12) is assumed to be of the form (outer expansion)

$$T(\rho, \mu) = F_0(P)\, T_0(\rho, \mu) + F_1(P)\, T_1(\rho, \mu) + F_2(P)\, T_2(\rho, \mu) + \dots , \tag{14}$$

where

$$F_{n+1}(P)/F_n(P) \to 0 \text{ as } P \to 0 . \tag{15}$$

The outer expansion is required to satisfy the boundary condition at infinity and, instead of satisfying the condition on the surface, to match with the inner expansion. The matching condition can be written as (Van Dyke [6])

$$\lim_{r\to\infty} t = \lim_{\rho\to 0} T \tag{16}$$

for $P \to 0$.

3. First Three Expansion Terms

The equation for t_0 is obtained by putting $P=0$ in eq. (6), that is, we have

$$\nabla_r^2 t_0 = 0 . \tag{17}$$

On inserting eqs. (13) and (14) into eq. (12), we have for T_0

$$\cos\theta \frac{\partial T_0}{\partial \rho} - \frac{\sin\theta}{\rho} \frac{\partial T_0}{\partial \theta} = \nabla_\rho^2 T_0 . \tag{18}$$

The required solutions of these two equations are already obtained by Acrivos and Taylor as follows.

$$t_0 = r^{-1} , \tag{19}$$

$$T_0 = \rho^{-1} \exp\{\tfrac{1}{2}\rho(\mu - 1)\} \tag{20}$$

and

$$F_0(P) = P . \tag{21}$$

From eq. (20), we have for small ρ

$$T \sim P\{\rho^{-1} + \tfrac{1}{2}(\mu - 1) + O(\rho)\} , \tag{22}$$

and hence, from the matching condition (16), we can expect for large r

$$f_1 t_1 \sim -\tfrac{1}{2} P(1 - \mu) . \tag{23}$$

In view of this, we have

$$f_1(P) = P , \tag{24}$$

and the equation for t_1 becomes

$$\nabla_r^2 t_1 = (-r^{-2} + r^{-5})\mu . \tag{25}$$

Here we have used the lower order approximation to calculate the convection term. The general solution of the equation is

$$t_1 = (\tfrac{1}{2} + \tfrac{1}{4} r^{-3})\mu + \sum_{n=0}^{\infty} (A_n r^n + B_n r^{-n-1}) P_n(\mu) , \tag{26}$$

where $P_n(\mu)$ is the Legendre polynomial of degree n and A_n, B_n are integral constants. Using the matching condition (23) and the condition $t_1=0$ at $r=1$, we have

$$\left.\begin{aligned} &A_0=-\tfrac{1}{2}, \quad A_n=0 \text{ for } n\geqq 1, \\ &B_0=\tfrac{1}{2}, \quad B_1=-\tfrac{3}{4}, \quad B_n=0 \text{ for } n\geqq 2. \end{aligned}\right\} \tag{27}$$

Therefore t_1 has been determined as

$$t_1=\tfrac{1}{2}(-1+r^{-1})+\tfrac{1}{2}(1-\tfrac{3}{2}r^{-2}+\tfrac{1}{2}r^{-3})P_1(\mu). \tag{28}$$

From eqs. (19) and (28), the asymptotic behavior of t for $r\to\infty$ is found to be

$$t\sim r^{-1}+P\left(\frac{\mu-1}{2}+\tfrac{1}{2}r^{-1}-\frac{3\mu}{4}r^{-2}+\ldots\right). \tag{29}$$

Matching consideration gives, for small ρ,

$$F_1T_1\sim\tfrac{1}{2}P^2\rho^{-1}. \tag{30}$$

In view of this, we have

$$F_1(P)=P^2, \tag{31}$$

and the equation for T_1 becomes

$$\cos\theta\frac{\partial T_1}{\partial\rho}-\frac{\sin\theta}{\rho}\frac{\partial T_1}{\partial\theta}=\nabla_\rho^2T_1. \tag{32}$$

This equation is identical with that for T_0 and the required solution is clearly

$$T_1=\tfrac{1}{2}\rho^{-1}\exp\{\tfrac{1}{2}\rho(\mu-1)\}. \tag{33}$$

From the eqs. (20), (21), (31) and (33), the matching condition for the third term of the inner expansion is found by usual procedure to be

$$f_2t_2\sim P^2\left\{\frac{(1-\mu)^2}{8}r-\frac{1-\mu}{4}+\ldots\right\}, \tag{34}$$

for large r. From this, we have

$$f_2(P)=P^2 \tag{35}$$

and the equation for t_2 becomes

$$\begin{aligned}\nabla_r^2t_2=&\tfrac{1}{3}r^{-1}+\tfrac{1}{12}r^{-4}-\tfrac{3}{4}r^{-6}+\tfrac{1}{3}r^{-7}-\tfrac{1}{2}(r^{-2}-r^{-5})P_1(\mu)\\&+(-\tfrac{1}{3}r^{-1}+\tfrac{3}{2}r^{-3}-\tfrac{5}{6}r^{-4}-\tfrac{3}{4}r^{-6}+\tfrac{5}{12}r^{-7})P_2(\mu).\end{aligned} \tag{36}$$

After a straightforward calculation, the required solution is found to be

$$\begin{aligned}t_2=&\tfrac{1}{4}(\tfrac{2}{3}r-1+\tfrac{7}{20}r^{-1}+\tfrac{1}{6}r^{-2}-\tfrac{1}{4}r^{-4}+\tfrac{1}{15}r^{-5})+\tfrac{1}{4}(-r+1-\tfrac{1}{2}r^{-2}+\tfrac{1}{2}r^{-3})P_1(\mu)\\&+\tfrac{1}{4}(\tfrac{1}{3}r-r^{-1}+\tfrac{5}{6}r^{-2}+\tfrac{3}{14}r^{-3}-\tfrac{1}{2}r^{-4}+\tfrac{5}{42}r^{-5})P_2(\mu).\end{aligned} \tag{37}$$

From the eqs. (19), (28) and (37), the matching condition for T_2 is easily found to be

$$F_2T_2\sim\tfrac{1}{4}P^3[-3\mu\rho^{-2}+\{\tfrac{7}{20}-P_2(\mu)\}\rho^{-1}+\ldots], \tag{38}$$

for small ρ. From this, we have

$$F_2(P)=P^3, \tag{39}$$

and the equation for T_2 is found to be identical with that for T_0. The general solution which vanishes at infinity is [1]

$$T_2=\pi^{\frac{1}{2}}\exp\{\tfrac{1}{2}\rho(\mu-1)\}\sum_{n=0}^{\infty}(2n+1)C_nK_{n+\frac{1}{2}}\left(\frac{\rho}{2}\right)P_n(\mu), \tag{40}$$

where C_n is an integral constant and $K_{n+\frac{1}{2}}$ is a modified Bessel function. The matching condition (38) determines C_n as

$$\left.\begin{aligned} C_0 &= \frac{17}{80\pi}, \\ C_1 &= -\frac{1}{8\pi}, \\ C_n &= 0 \text{ for } n \geqq 2 \end{aligned}\right\} \tag{41}$$

and we have

$$T_2 = -\tfrac{1}{8}\rho^{-1}\{\tfrac{17}{10}-3(1+2\rho^{-1})\mu\}\exp\{\tfrac{1}{2}\rho(\mu-1)\}. \tag{42}$$

4. Higher Expansion Terms

By repeating the similar procedure to that used so far, we can also obtain the higher order expansion terms. For fourth term of the inner expansion, we can finally obtain, after a straightforward calculation

$$f_3(P) = P^3 \tag{43}$$

$$\begin{aligned} t_3 = {} & \tfrac{1}{12}(-\tfrac{1}{2}r^2+r-\tfrac{51}{40}+\tfrac{1}{20}r^{-1}+\tfrac{1}{4}r^{-2}-\tfrac{1}{6}r^{-4}+\tfrac{1}{10}r^{-5}) \\ & +\tfrac{1}{8}(\tfrac{3}{5}r^2-r+\tfrac{23}{20}+\tfrac{3}{5}r^{-1}-\tfrac{247}{140}r^{-2}+\tfrac{11}{40}r^{-3}+\tfrac{3}{14}r^{-4} \\ & +\tfrac{3}{70}r^{-5}-\tfrac{3}{20}r^{-6}+\tfrac{9}{280}r^{-7})P_1(\mu) \\ & +\tfrac{1}{24}(-r^2+r-2r^{-1}+\tfrac{5}{2}r^{-2}-\tfrac{5}{14}r^{-3}-\tfrac{1}{2}r^{-4}+\tfrac{5}{14}r^{-5})P_2(\mu) \\ & +\tfrac{3}{20}(\tfrac{1}{18}r^2-\tfrac{1}{4}+\tfrac{1}{3}r^{-1}+\tfrac{3}{28}r^{-2}-\tfrac{1}{2}r^{-3}+\tfrac{15}{56}r^{-4}+\tfrac{3}{56}r^{-5}-\tfrac{1}{12}r^{-6}+\tfrac{1}{63}r^{-7})P_3(\mu). \end{aligned} \tag{44}$$

From the forms of t_0, t_1, t_2 and t_3, the matching condition for $F_3 T_3$ is found to be

$$\begin{aligned} F_3 T_3 \sim {} & \tfrac{1}{4}P^4[P_1(\mu)\rho^{-3}+\tfrac{1}{2}\{\tfrac{1}{3}-P_1(\mu)+\tfrac{5}{3}P_2(\mu)\}\rho^{-2} \\ & +\{\tfrac{11}{60}+\tfrac{3}{10}P_1(\mu)-\tfrac{1}{3}P_2(\mu)+\tfrac{1}{5}P_3(\mu)\}\rho^{-1}+\ldots], \end{aligned} \tag{45}$$

for small ρ. In view of this, we have

$$F_3(P) = P^4, \tag{46}$$

and the equation for T_3 becomes

$$\nabla_\rho^2 T_3 - \cos\theta\frac{\partial T_3}{\partial \rho} + \frac{\sin\theta}{\rho}\frac{\partial T_3}{\partial \theta} = -\frac{\cos\theta}{\rho^3}\frac{\partial T_0}{\partial \rho} - \frac{\sin\theta}{2\rho^4}\frac{\partial T_0}{\partial \theta}. \tag{47}$$

The transformation

$$T_3(\rho,\mu) = T_3^*(\rho,\mu)\exp(\tfrac{1}{2}\rho\mu) \tag{48}$$

reduces the eq. (47) to

$$(\nabla_\rho^2-\tfrac{1}{4})T_3^* = \{(\rho^{-5}+\tfrac{1}{2}\rho^{-4})P_1(\mu)-\tfrac{1}{2}\rho^{-4}P_2(\mu)\}\exp(-\tfrac{1}{2}\rho), \tag{49}$$

and the general solution that vanishes at infinity is easily found to be

$$\begin{aligned} T_3 = \Big[& D_0\rho^{-1}+(\tfrac{1}{4}\rho^{-3}+2D_1\rho^{-2}+D_1\rho^{-1})P_1(\mu) \\ & +\{(12D_2-\tfrac{1}{4})\rho^{-3}+6D_2\rho^{-2}+D_2\rho^{-1}\}P_2(\mu) \\ & +\sum_{n=3}^{\infty} D_n\left\{\sum_{m=0}^{n}\frac{(n+m)!}{m!(n-m)!\rho^{m+1}}\right\}P_n(\mu)\Big]\exp\{\tfrac{1}{2}\rho(\mu-1)\}, \end{aligned} \tag{50}$$

where D_n is an integral constant. Matching this solution with $t(r,\mu)$, we find that

$$\left.\begin{aligned} D_0 &= \tfrac{1}{15}\,, \\ D_1 &= 0\,, \\ D_2 &= \tfrac{1}{48}\,, \\ D_n &= 0 \text{ for } n \geqq 3\,, \end{aligned}\right\} \tag{51}$$

and therefore T_3 has been determined completely as follows.

$$T_3 = \{\tfrac{1}{15}\rho^{-1} + \tfrac{1}{4}\rho^{-3} P_1(\mu) + \tfrac{1}{8}(\rho^{-2} + \tfrac{1}{6}\rho^{-1}) P_2(\mu)\} \exp\{\tfrac{1}{2}\rho(\mu-1)\}\,. \tag{52}$$

Finally, the fifth term of the inner expansion is found, after a straightforward but somewhat tedious calculation, to be

$$f_4(P) = P^4\,, \tag{53}$$

$$\begin{aligned} t_4 = {} & \tfrac{1}{240}(2r^3 - 5r^2 + \tfrac{23}{2}r - 6 - \tfrac{18447}{3920}r^{-1} + \tfrac{35}{8}r^{-2} + \tfrac{9}{7}r^{-3} \\ & - \tfrac{253}{56}r^{-4} + \tfrac{17}{20}r^{-5} + \tfrac{17}{56}r^{-6} + \tfrac{3}{49}r^{-7} - \tfrac{3}{16}r^{-8} + \tfrac{1}{28}r^{-9}) \\ & + \tfrac{1}{16}(-\tfrac{4}{15}r^3 + \tfrac{3}{5}r^2 - \tfrac{27}{20}r + \tfrac{1}{6} + \tfrac{3}{5}r^{-1} + \tfrac{361}{1680}r^{-2} + \tfrac{1}{20}r^{-3} \\ & + \tfrac{3}{112}r^{-4} - \tfrac{1}{42}r^{-5} - \tfrac{1}{20}r^{-6} + \tfrac{9}{280}r^{-7}) P_1(\mu) \\ & + \tfrac{1}{112}(\tfrac{4}{3}r^3 - \tfrac{7}{3}r^2 + \tfrac{269}{60}r + 3 - \tfrac{2273}{210}r^{-1} + \tfrac{672}{96}r^{-2} + \tfrac{2677}{3080}r^{-3} \\ & - \tfrac{61}{15}r^{-4} - \tfrac{1}{24}r^{-5} + \tfrac{277}{420}r^{-6} + \tfrac{1}{8}r^{-7} - \tfrac{1}{4}r^{-8} + \tfrac{139}{3080}r^{-9}) P_2(\mu) \\ & + \tfrac{1}{8}(-\tfrac{1}{30}r^3 + \tfrac{1}{30}r^2 - \tfrac{3}{20} + \tfrac{1}{5}r^{-1} - \tfrac{1}{28}r^{-2} - \tfrac{1}{10}r^{-3} + \tfrac{67}{504}r^{-4} \\ & - \tfrac{1}{56}r^{-5} - \tfrac{1}{60}r^{-6} + \tfrac{1}{105}r^{-7}) P_3(\mu) + \tfrac{1}{560}(\tfrac{1}{3}r^3 - 2r \\ & + \tfrac{11}{3} + \tfrac{9}{7}r^{-1} - 9r^{-2} + 7r^{-3} + \tfrac{9}{4}r^{-4} - \tfrac{23935}{4004}r^{-5} \\ & + \tfrac{103}{42}r^{-6} + \tfrac{9}{22}r^{-7} - \tfrac{1}{2}r^{-8} + \tfrac{22}{273}r^{-9}) P_4(\mu)\,. \end{aligned} \tag{54}$$

5. Conclusion

From the inner expansion solution, an expression for the average Nusselt number defined by

$$N = -\int_{-1}^{1} \left(\frac{\partial t}{\partial r}\right)_{r=1} d\mu\,, \tag{55}$$

is found to be

$$N = 2 + P - \tfrac{13}{40}P^2 + \tfrac{7}{40}P^3 - \tfrac{48407}{470400}P^4 + \dots\,. \tag{56}$$

A comparison of the result with that of Acrivos and Taylor is shown in fig. 2. At extremely small values for P, both results show good agreement. This behaviour is consistent with the

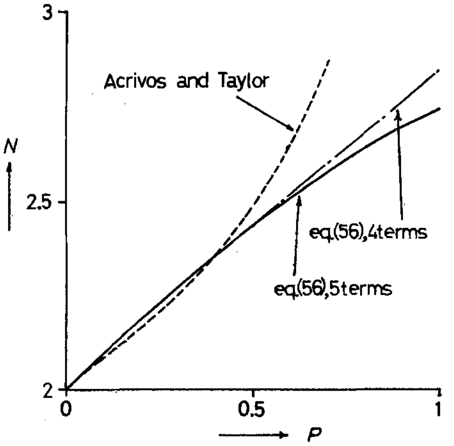

Figure 2.

statement mentioned earlier about the applicability of Acrivos and Taylor's results to low Prandtl number fluids. For $P > 0.5$, the deviation of Acrivos and Taylor's curve from that of the present theory rapidly increases.

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